NOTES ON NON-INTERFERING ELECTRIC AND MAGNETIC FIELDS

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ABSTRACT. It is shown that steady magnetic fields and time-dependent electric fields may exist simultaneously without mutual interference. This happens when the time derivative of the current is irrotational. In this case Maxwell's equation may be broken up into two sets of equations containing electric and magnetic quantities separately.

INTRODUCTION

It follows, in general, from the Maxwell electro-magnetic field equations that the electric and magnetic field quantities mutually interfere each other. Since the terms with time derivatives in Maxwell's equation is the nexus between the electric and magnetic field vectors, it is quite evident that the electric and magnetic field quantities do not interfer each other when they are steady, i.e. independent of time. However, a critical examination of the Maxwell equation reveals that it is not always necessary that the field quantities should not vary with time for the existence of mutually non-interfering electric and magnetic fields. The object of this note is to examine the possibility of non-interfering electric and magnetic fields in general and to investigate the nature of this kind of field. Without going into details one can conclude from Faraday's law of induction that the electric and magnetic fields may exist without interfering with each other only when the magnetic field is steady and hence the electric vector should be irrotational; though, the latter may be varying with time.

In the next section, it is shown that the Maxwell field equations separate out into two sets containing only electric or magnetic quantities, when the time derivative of the current $\underline{j}$ is an irrotationzl vector, i.e.

$$\underline{\nabla} \times \frac{\partial \underline{j}}{\partial t} = 0. \qquad .. \quad (1)$$

It is further shown that any steady magnetic field may exists along with a class of electric field varying with time. The last section is devoted to the reduction of Maxwell's equations, in general, for which the given current satisfies equation (1).

THE SEPARATION OF ELECTRIC AND MAGNETIC FIELDS

The Maxwell field equations are given by

$$\underset{\sim}{\nabla}\times\underset{\sim}{H}-\frac{\epsilon}{c}\frac{\partial \underset{\sim}{E}}{\partial t}=\underset{\sim}{j} \quad .. \quad (2)$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{E}+\frac{\mu}{c}\frac{\partial \underset{\sim}{H}}{\partial t}=0 \quad .. \quad (3)$$

$$\epsilon\underset{\sim}{\nabla}.\underset{\sim}{E}=q \quad .. \quad (4)$$

$$\mu\underset{\sim}{\nabla}.\underset{\sim}{H}=0. \quad .. \quad (5)$$

In order that $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are not dependent on each other, it is quite evident from equation (3) that each of the terms should be zero separately. Thus it is necessary that

$$\frac{\partial \underset{\sim}{H}}{\partial t}=0 \quad .. \quad (6)$$

and

$$\underset{\sim}{\nabla}\times\underset{\sim}{E}=0. \quad .. \quad (7)$$

$\underset{\sim}{H}$ should be independent of time and $\underset{\sim}{E}$ should be irrotational. Next step is to separate $\underset{\sim}{E}$ and $\underset{\sim}{H}$ terms in eq. (2). To accomplish this, one observes that any vector may be separated in two parts, one of which is irrotational and the other is divergence-free. Thus

$$\underset{\sim}{j}=\underset{\sim}{\nabla}\chi+\underset{\sim}{\nabla}\times\underset{\sim}{J} \quad .. \quad (8)$$

with

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{J}=0, \quad .. \quad (9)$$

and χ any function of space-time. Because of the continuity equation relating the charge and current

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{j}+\frac{1}{c}\frac{\partial q}{\partial t}=0, \quad (10)$$

the equation for χ is

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\chi+\frac{1}{c}\frac{\partial q}{\partial t}=0. \quad (11)$$

From eqs. (4), (7), and (11), it follows that

$$\underset{\sim}{\nabla}\cdot\left(\frac{c}{c}\;\frac{\partial \underset{\sim}{\mathrm{E}}}{\partial t}+\underset{\sim}{\nabla}\chi\right)=0 \qquad \text{.. (12)}$$

Now, let us note that the separation of $\underset{\sim}{j}$ in equation (8) is not unique as one can write $\chi' = \chi+\chi_0$ for χ and $\mathrm{J}' = \mathrm{J}+J_0$ for $\underset{\sim}{J}$, such that

$$\underset{\sim}{\nabla}\chi_0 = \underset{\sim}{\nabla}\times \underset{\sim}{J}_0 \qquad \text{.. (13)}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\chi_0 = 0 \qquad \text{.. (14)}$$

$$\underset{\sim}{\nabla}\cdot J_0 = 0. \qquad \text{.. (15)}$$

Obviously such χ_0, J_0's exist. Hence with suitable choice of χ_0 we can write from eq. (12)

$$\frac{c}{c}\;\frac{\partial \underset{\sim}{\mathrm{E}}}{\partial t}+\underset{\sim}{\nabla}\chi = 0. \qquad \text{.. (16)}$$

This is due to the fact that the left hand side of eq. (16) being irrotational the right hand side is a divergence-free gradient vector which can be absorved in $\underset{\sim}{\nabla}\chi$. If this is the case equation (2) will then reduces to

$$\nabla\times\mathbf{H} = \nabla\times J. \qquad \text{.. (17)}$$

Since H is independent of time

$$\frac{\partial}{\partial t}\,\underset{\sim}{\nabla}\times \underset{\sim}{J} = 0. \qquad \text{.. (18)}$$

Hence equation (8) leads to

$$\frac{\partial}{\partial t}\,\underset{\sim}{\nabla}\times \underset{\sim}{J} = 0. \qquad \text{.. (1)}$$

Thus, in order that $\underset{\sim}{\mathrm{E}}$ and $\underset{\sim}{\mathrm{H}}$ may exist without interfering with each other, it is necessary that $\dot{\mathrm{j}}$ should satisfy equation (1) as noted in the introduction. In this case, the Maxwell field equations separate out in two groups as

$$\left.\begin{aligned} &\underset{\sim}{\nabla}\times\underset{\sim}{\mathbf{E}} = 0\\ &\epsilon\;\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = q(\underset{\sim}{r}, t)\\ &-\frac{\epsilon}{c}\;\frac{\partial \underset{\sim}{\mathrm{E}}}{\partial t} = \underset{\sim}{\nabla}\chi(\underset{\sim}{r}, t), \end{aligned}\right\} \qquad \text{.. (19)}$$

and

$$\left.\begin{aligned} \nabla\times\mathbf{H} &= \nabla\times\mathbf{J}(r) \\ \nabla\cdot\mathbf{H} &= 0 \\ \frac{\partial\mathbf{H}}{\partial t} &= 0 \end{aligned}\right\} \tag{20}$$

THE REDUCTION OF MAXWELL'S EQATIONS

In this section we will determine $\mathbf{E}$ and $\mathbf{H}$ for the class of charges and currents which satisfy equation (1). Given the current $\mathbf{j}$, from equations. (8). (9) and (20), H is determined by

$$\mathbf{H} = \mathbf{J}$$

and

$$\left.(\nabla\cdot\nabla)\mathbf{J} = -\nabla\times\mathbf{j}.\right\} \tag{21}$$

Next $\mathbf{E}$ is obtained from eq. (18) by

$$\mathbf{E} = -\nabla\phi$$

and

$$\left.\nabla\cdot\nabla\phi = -q/\epsilon.\right\} \tag{22}$$

The third member of the set of equation (19) is redundant in virtue of equation (11). It is not irrelevant to mention that the most general solution of $\mathbf{E}$, $\mathbf{H}$ are determined from these by adding the solution of the homogeneous equations, which are obtained by putting the right hand sides of equations (19) and (20) as zero. The latter's contributions are to be determined by the relevant boundary conditions.

Finally, it is to be noted that even with j satisfying eq. (1), the electric and magnetic field quantities may not be of non-interfering nature, in general. This may happen if the boundary condition is not consistence with the nature of the field envisaged here. However, in such cases it is always possible to reduced the field equation to homogeneous one, i.e. to source free field equations. Let $\mathbf{E}$ and $\mathbf{H}$ be any solution satisfying the inhomogeneous equations. (19) and (20), as mentioned above. Further let us write

$$\mathbf{E} = \mathbf{E}'+\mathbf{E}_0 \tag{23}$$

and

$$\mathbf{H} \quad \mathbf{H}'+\mathbf{H}_0. \tag{24}$$

Substituting these in eqs. (2)–(5), we can find the equation for $\underset{\sim}{\mathbf{E}}_0$, and $\underset{\sim}{\mathbf{H}}_0$. Since $\underset{\sim}{\mathbf{E}}'$ and $\underset{\sim}{\mathbf{H}}'$ satisfy equations (19) and (20) respectively one obtains

$$\nabla \times \underset{\sim}{\mathbf{H}}_0 - \frac{\epsilon}{c} \frac{\partial \underset{\sim}{\mathbf{E}}_0}{\partial t} = 0 \qquad \text{.. } (2')$$

$$\nabla \times \underset{\sim}{\mathbf{E}}_0 + \frac{\mu}{c} \frac{\partial \underset{\sim}{\mathbf{H}}_0}{\partial t} = 0 \qquad \text{.. } (3')$$

$$\nabla \cdot \underset{\sim}{E}_0 = 0 \qquad \text{.. } (4')$$

$$\nabla \cdot H_0 = 0. \qquad \text{.. } (5')$$

Thus $\underset{\sim}{\mathbf{E}}_0$ and $\underset{\sim}{\mathbf{H}}_0$ satisfy the homogeneous field equations. Their role is to satisfy the boundary condition. The point to be emphasized is that the part of the field quantity which depends on the source is always an electric field varying with time and a steady magnetic field whenever the time derivative of the current is irrotational.